General combinatorical structure of truth tables of bracketed formulae connected by implication.

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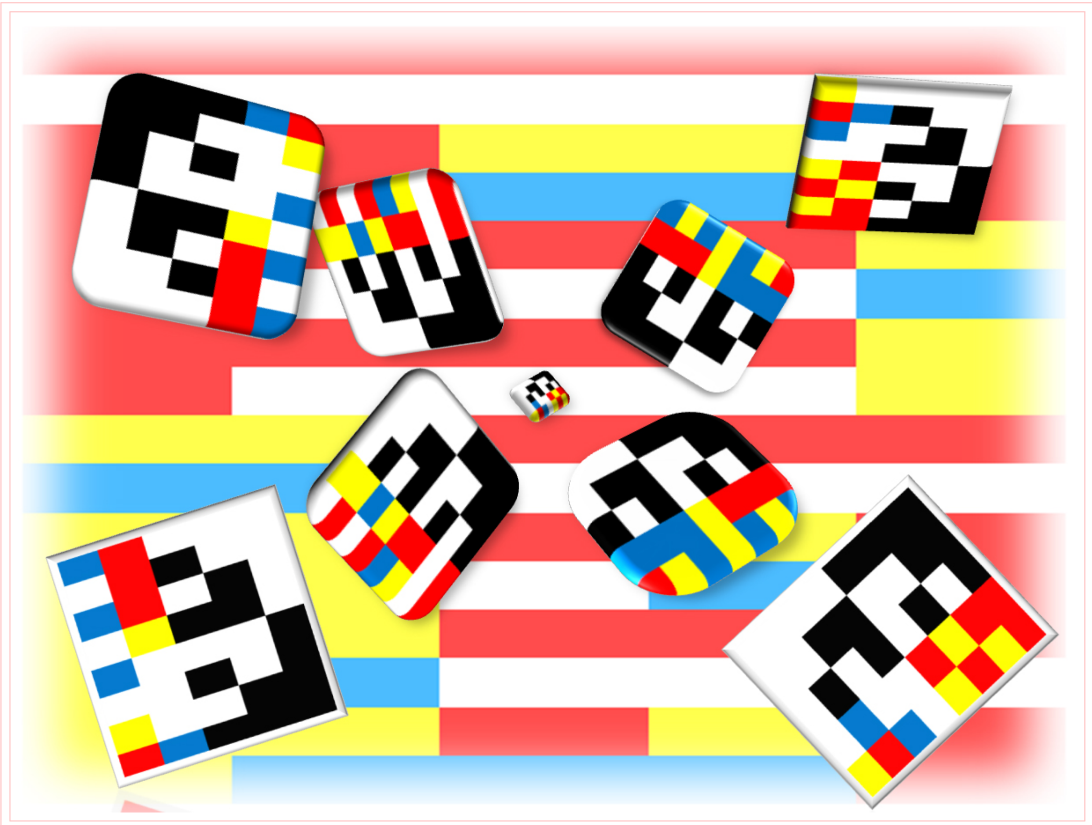

Abstract

In this paper we investigate the general combinatorical structure of the truth tables of all bracketed formulae with n distinct variables connected by the binary connective of implication, an m-implication.

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Life is a tragedy when seen in close-up, but a comedy in long-shot.

C. Chaplin

Contents

1 Preface **3**

2 Introduction **4**

3 General Structure **4**

4 Counting true entries in truth tables of bracketed formulae connected by implication **5**

4.1 Case 4 . . . 5
4.2 Case 3 . . . 6
4.3 Case 2 . . . 7
4.4 Case 1 . . . 8

5 Asymptotic Estimate 1 **11**

6 Counting true entries in truth tables of bracketed formulae connected by modified implication **13**

6.1 Type 1 . . . 13
6.1.1 Case 4 . . . 14
6.1.2 Case 3 . . . 15
6.1.3 Case 2 . . . 16
6.1.4 Case 1 . . . 16
6.2 Type 2 . . . 17
6.2.1 Case 4 . . . 18
6.2.2 Case 3 . . . 18
6.2.3 Case 2 . . . 19
6.2.4 Case 1 . . . 20
6.3 Type 3 . . . 21

7 Asymptotic Estimate 2 **22**

7.1 Asymptotic for Type 1 and 3 . . . 22
7.2 Asymptotic for Type 2 . . . 24

8 Parity **25**

Notation

(1) $p_1, p_2, \ldots, p_n$ and ϕ, ψ are propositional variables.

(2) 'True' will be denoted by 1

(3) 'False' will be denoted by 0

(4) Set of counting numbers is denoted by $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

(5) Set of even counting numbers is denoted by $\mathbb{E}$

(6) Set of odd counting numbers is denoted by $\mathbb{O}$

(7) : such that

(8) ν is the valuation function : $\nu(\phi) = 1$ if ϕ is true, and $\nu(\phi) = 0$ if ϕ is false.

(9) $\wedge$ the conjunction

(10) $\vee$ the disjunction

(11) $\rightarrow$ the binary connective of implication

(12) $\neg$ the negation operator

(13) $\#c$ denotes the case number in $t_n^{\#c}$, where $c = 1, 2, 3$.

(14) $\star$ is for QED.

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1 Preface

This project begun with the following question: *Can we count the number of true entries in truth tables of bracketed formulae connected by implication, and modified-implication rules?* The primitive answer is to get the number of false entries then subtract it from the total number of entries. But this calculation is not precise enough to see the fruitfulness of the truth tables of these kind. In this project we have underlined that 'unlike the false entries in the truth tables that are connected by the binary connective of implication or by m-implication, true entries are not homogeneous structures'. The cover picture on the title page is designed to give an intuitive background for this inhomogeneous structure.

By reading this project the reader will encounter eight Catalan like sequences, and their asymptotics. We have started this project by counting the total number of entries (or rows), in all truth tables for bracketed implication. Then in section 3, we used the recurrence relation of the total number of rows and we worked backwards to find out the general structure of the truth tables. Alas, we showed that the structure of the recurrence relation of the true, or false entries is inherited from the structure of the recurrence relation of the total number of entries. In section 4 and 6 we counted the true entries first in using ordinary implication, and later in modified binary connective of implication. In section 5 and 7 we dealt with asymptotic of the sequences that we have seen in section 4 and 6 respectively. In the last section we showed that the sequences that are in section 4 and 6 preserve the parity of Catalan numbers.

Volkan Yildiz,
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2 Introduction

In [2] and [7] we have mentioned that the total number of rows in all truth tables of bracketed implications with n distinct variables is g_n and it has the following generating function and the explicit formula respectively:

$$G(x) = \frac{1-\sqrt{1-8x}}{2}, \qquad \textit{and} \qquad g_n = 2^n C_n,$$

$$\textit{where } C_n \textit{ is the nth Catalan number and} \qquad C_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

By using the explicit formula it is straightforward to calculate the values of g_n. The table below illustrates this up to $n = 12$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
g_n	0	2	4	16	80	448	2688	16896	109824	732160	4978688	34398208	240787456

Proposition 2.1 *Let g_n be the total number of rows in all truth tables for bracketed implication with n distinct variables $p_1, \ldots, p_n$. Then*

$$\boxed{g_n = \sum_{i=1}^{n-1} g_i g_{n-i}, \qquad \textit{with} \quad g_1 = 2. \qquad (1)}$$

Proof [First Proof]
For $p_1 \to \ldots \to p_n$ there are 2^n rows and C_n columns, hence there are $2^n C_n$ rows altogether.

$$g_n = 2^n C_n = 2^n \sum_{i=1}^{n-1} C_i C_{n-i} = \sum_{i=1}^{n-1} (2^i C_i)(2^{n-i} C_{n-i}) = \sum_{i=1}^{n-i} g_i g_{n-i}, \quad \textit{with } g_1 = 2. \ \star$$

[Second proof, 'intuitive']

Consider n distinct propositions $p_1, \ldots, p_n$, for $i \geq 1$:

$$\underbrace{p_1 \to \ldots \to p_i}_{g_i rows} \to \underbrace{p_{i+1} \to \ldots \to p_n}_{g_{n-i} rows}$$

There are g_i rows for $p_1 \to \ldots \to p_i$, and there are g_{n-i} rows for $p_{i+1} \to \ldots \to p_n$, and the number of choices is given the recurrence relation (1) . $\star$

3 General Structure

To see the combinatorical structure of the truth tables of bracketed formulae connected by implication, we need to make use of the method of working backwards. This method allows us to avoid unnecessary choices altogether. In ancient Greek it was mentioned by the mathematician Pappus and in recent times the method of working backwards has been discussed G. Polya. Recall that

$$g_n = \sum_{i=1}^{n-1} g_i g_{n-i}$$

Since each table consists of false and true rows, we let $g_i = t_i + f_i$, where t_i, f_i is the corresponding number of 'true', 'false' rows in g_i respectively. Then

$$g_n = \sum_{i=1}^{n-1} (t_i + f_i)(t_{n-i} + f_{n-i})$$

If we expand the right hand side:

$$g_n = \underbrace{\sum_{i=1}^{n-1} t_i t_{n-i}}_{\textit{case 1}} + \underbrace{\sum_{i=1}^{n-1} f_i t_{n-i}}_{\textit{case 2}} + \underbrace{\sum_{i=1}^{n-1} f_i f_{n-i}}_{\textit{case 3}} + \underbrace{\sum_{i=1}^{n-1} t_i f_{n-i}}_{\textit{case 4}} \tag{2}$$

We can now partition the right hand side into more tangible cases and explain what each of these summands mean. Let ψ and ϕ be propositional variables, and let ν be the *valuation* function, then

$$\nu(\psi \to \phi) = 1 \iff (\nu(\psi) = 0 \vee \nu(\phi) = 1).$$

Therefore there are three cases to consider here:

$$\underbrace{(\nu(\psi) = 1 \wedge \nu(\phi) = 1)}_{\textit{case 1}} \vee \underbrace{(\nu(\psi) = 0 \wedge \nu(\phi) = 1)}_{\textit{case 2}} \vee \underbrace{(\nu(\psi) = 0 \wedge \nu(\phi) = 0)}_{\textit{case 3}}$$

Addition to the three cases in above there is also the fourth case, known as the *'disastrous combination'*:

$$\nu(\psi \to \phi) = 0 \iff \underbrace{(\nu(\psi) = 1 \wedge \nu(\phi) = 0)}_{\textit{case 4}}.$$

The four cases in equation (2) coincide with the four cases in the penultimate lines respectively, and we get the following theorem:

Theorem 3.1 *Let g_n be the total number of rows in all truth tables of bracketed implications with n distinct variables. Then for $n \geq 2$, g_n can be partition into four cases as below*

$$g_n = t_n^{\#1} + t_n^{\#2} + t_n^{\#3} + f_n, \ \textit{where}$$

$$t_n^{\#1} = \sum_{i=1}^{n-1} t_i t_{n-i}, \quad t_n^{\#2} = \sum_{i=1}^{n-1} f_i t_{n-i}, \quad t_n^{\#3} = \sum_{i=1}^{n-1} f_i f_{n-i}, \ \textit{and} \quad f_n = \sum_{i=1}^{n-1} t_i f_{n-i}.$$

with

$$0 = t_1^{\#1} = t_1^{\#2} = t_1^{\#3}, \quad \textit{and} \quad f_1 = 1.$$

In coming parts of this paper we will investigate each of the above cases in more detail.

4 Counting true entries in truth tables of bracketed formulae connected by implication

4.1 Case 4

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \to \phi) = 0 \quad : \quad (\nu(\psi) = 1 \wedge \nu(\phi) = 0).$$

This case has been discussed already in [2]; we had the following results.

Theorem 4.1 *Let f_n be the number of rows with the value "false" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \dots, p_n$ connected by the binary connective of implication. Then*

$$\boxed{f_n = \sum_{i=1}^{n-1} (2^i C_i - f_i) f_{n-i}, \quad \textit{with} \;\; f_1 = 1 \quad (3)}$$

and for large n, $f_n \sim \left(\frac{3-\sqrt{3}}{6}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}$. *Where C_i is the ith Catalan number.*

If we look closely to the recurrence relation (3), since $t_i = (2^i C_i - f_i)$ we obtain the 'case 4' in equation (3):

$$f_n = \sum_{i=1}^{n-1} (2^i C_i - f_i) f_{n-i} = \sum_{i=1}^{n-1} t_i f_{n-i}, \quad \textit{with} \;\; f_1 = 1.$$

First ten terms of $\{f_n\}_{n>0}$ are,

$$1, 1, 4, 19, 104, 614, 3816, 24595, 162896, 1101922, \dots$$

We have also shown in [2] that f_n has the following generating function:

$$\boxed{F(x) = \frac{-1 - \sqrt{1-8x} + \sqrt{2 + 2\sqrt{1-8x} + 8x}}{4}.}$$

We have also shown in [8] that the sequence $\{f_n\}_{n \geq 1}$ preserves the parity of Catalan numbers.

4.2 Case 3

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \to \phi) = 1 \quad : \quad (\nu(\psi) = 0 = \nu(\phi)).$$

In this case we are interested in formulae obtained from $p_1 \to \dots \to p_n$ by inserting brackets so that the valuation of the first i bracketing and the rest $(n-i)$ bracketing both give 0, 'false'. The table 1 below shows the truth tables, (merged into one), for the two bracketed implications in $n = 3$ variables. The corresponding case 3 truth values are denoted in green.

Proposition 4.2 *Let $t_n^{\#3}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \dots, p_n$ connected by the binary connective of implication such that the valuation of the first i bracketing and the rest $(n-i)$ bracketing both give 0, 'false'. Then*

$$t_n^{\#3} = \sum_{i=1}^{n-1} f_i f_{n-i} \quad \textit{where } \{f_i\}_{i \geq 1} \textit{ is the sequence in equation (3).} \tag{4}$$

Proof A row with the value 1, 'true', comes from an expression $\psi \to \phi$, where $\nu(\psi) = 0$ and $\nu(\phi) = 0$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Table 1: $n = 3$

p_1	p_2	p_3	$p_1 \to (p_2 \to p_3)$	$(p_1 \to p_2) \to p_3$
1	1	1	1	1
1	1	0	0	0
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	1	0
0	0	1	1	1
0	0	0	1	0

It is now very easy to find out the generating function of $t_n^{\#3}$ by using the relation (4). Let $T_3(x) = \sum_{n\geq 1} t_n^{\#3} x^n$ then $T_3(x) = F(x)^2$, and hence we get the following proposition:

Proposition 4.3 *The generating function for the sequence* $\{t_n^{\#3}\}_{n\geq 0}$ *is given by*

$$\boxed{T_3(x) = \frac{2 + 2\sqrt{1-8x} - \sqrt{2+2\sqrt{1-8x}+8x} - \sqrt{1-8x}\sqrt{2+2\sqrt{1-8x}+8x}}{8}.}$$

By using Maple we find the first 21 terms of this sequence:

$$\begin{aligned}\{t_n^{\#3}\}_{n>1} \quad = \quad & 1, 2, 9, 46, 262, 1588, 10053, 65686, 439658, 2999116,\\ & 20774154, 145726348, 1033125004, 7390626280, 53281906861,\\ & 386732675046, 2823690230850, 20725376703324,\\ & 152833785130398, 1131770853856100, 8412813651862868, \ldots\end{aligned}$$

We will discuss the asymptotic and number theoretical results in a separate chapter. Now we have to embark on case 2.

4.3 Case 2

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \to \phi) = 1 \quad : \quad (\nu(\psi) = 0 \ \wedge \ \nu(\phi) = 1).$$

In this case we are interested in formulae obtained from $p_1 \to \ldots \to p_n$ by inserting brackets so that the valuation of the first i bracketing give 0, 'false', and the rest $(n-i)$ bracketing give 1, 'true'.

Proposition 4.4 *Let $t_n^{\#2}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of implication such that the valuation of the first i bracketing is 0 and and the rest $(n-i)$ bracketing gives 1. Then,*

$$t_n^{\#2} = \sum_{i=1}^{n-1} f_i t_{n-i} \tag{5}$$

Proof A row with the value 1, 'true', comes from an expression $\psi \to \phi$, where $\nu(\psi) = 0$ and $\nu(\phi) = 1$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Table 2: $n = 3$

p_1	p_2	p_3	$p_1 \to (p_2 \to p_3)$	$(p_1 \to p_2) \to p_3$
1	1	1	1	1
1	1	0	0	0
1	0	1	1	**1**
1	0	0	1	1
0	1	1	**1**	1
0	1	0	1	0
0	0	1	**1**	1
0	0	0	**1**	0

The table 2 above, (merged into one), for the two bracketed implications in $n = 3$ variables, indicates the corresponding truth values in bold.

Corollary 4.5 *$t_n^{\#2} = f_n$ for $n \geq 2$.*

Proof Since $t_{n-i} = (2^{n-i}C_{n-i} - f_{n-i})$,

$$t_n^{\#2} = \sum_{i=1}^{n-1} f_i(2^{n-i}C_{n-i} - f_{n-i}) \tag{6}$$

More explicitly,

$$\begin{aligned} t_n^{\#2} &= f_1(2^{n-1}C_{n-1} - f_{n-1}) + \ldots + f_{n-2}(2^2C_2 - f_2) + f_{n-1}(2^1C_1 - f_1) \\ &= f_{n-1}(2^1C_1 - f_1) + f_{n-2}(2^2C_2 - f_2) + \ldots + f_1(2^{n-1}C_{n-1} - f_{n-1}) \\ &= \sum_{i=1}^{n-1}(2^iC_i - f_i)f_{n-i} \\ &= f_n. \end{aligned}$$

⋆

Thus for $n \geq 2$, Case 2 coincides with Case 4, which we have investigated in great detail in [2], and in [8].

Proposition 4.6 *The generating function for the sequence $\{t_n^{\#2}\}_{n\geq 0}$ is given by*

$$\boxed{T_2(x) = F(x) - x = \frac{-1 - \sqrt{1-8x} + \sqrt{2 + 2\sqrt{1-8x} + 8x} - 4x}{4}.}$$

4.4 Case 1

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \to \phi) = 1 \quad : \quad (\nu(\psi) = 1 = \nu(\phi)).$$

In this case we are interested in formulae obtained from $p_1 \to \ldots \to p_n$ by inserting brackets so that the valuation of the first i bracketing and the rest $(n - i)$ bracketing give 1, 'true'.

Table 3, (merged into one), for the two bracketed implications in $n = 3$ variables, indicates the corresponding truth values in red.

Table 3: $n = 3$

p_1	p_2	p_3	$p_1 \to (p_2 \to p_3)$	$(p_1 \to p_2) \to p_3$
1	1	1	**1**	**1**
1	1	0	0	0
1	0	1	**1**	1
1	0	0	**1**	1
0	1	1	1	**1**
0	1	0	1	0
0	0	1	1	**1**
0	0	0	1	0

Proposition 4.7 *Let $t_n^{\#1}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of implication such that the valuation of the first i bracketing and and the rest of $(n-i)$ bracketing gives 1. Then,*

$$t_n^{\#1} = \sum_{i=1}^{n-1} t_i t_{n-i} \tag{7}$$

Proof A row with the value 1, 'true', comes from an expression $\psi \to \phi$, where $\nu(\psi) = 1$ and $\nu(\phi) = 1$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Let $T_1(x) = \sum_{n\geq 1} t_n^{\#1} x^n$, since $G(x) = F(x) + T_1(x) + T_2(x) + T_3(x)$ then it is easy to find that $T_1(x)$ has the following explicit algebraic representation:

Proposition 4.8 *The generating function for the sequence $\{t_n^{\#1}\}_{n\geq 0}$ is given by*

$$\boxed{T_1(x) = \frac{6 - 2\sqrt{1-8x} - 3\sqrt{2+2\sqrt{1-8x}+8x} + \sqrt{1-8x}\sqrt{2+2\sqrt{1-8x}+8x}}{8}.}$$

By using Maple we find the first 21 terms of this sequence:

$$\begin{aligned}\{t_n^{\#1}\}_{n\geq 2} &= 1, 6, 33, 194, 1198, 7676, 50581, 340682, 2335186, 16237284,\\ &\quad 114255994, 812107412, 5822171548, 42052209400, 305714145869,\\ &\quad 2235262899418, 16426616425002, 121265916776148,\\ &\quad 898878250833358, 6687497426512700, 49920590244564484, \ldots\end{aligned}$$

The below table shows the sequences which we have discussed so far, up to $n = 11$.

n	0	1	2	3	4	5	6	7	8	9	10	11
2^n	1	2	4	8	16	32	64	128	256	512	1024	2048
C_n	0	1	1	2	5	14	42	132	429	1430	4862	16796
g_n	0	2	4	16	80	448	2688	16896	109824	732160	4978688	34398208
f_n	0	1	1	4	19	104	614	3816	24595	162896	1101922	7580904
$t_n^{\#1}$	0	0	1	6	33	194	1198	7676	50581	340682	2335186	16237284
$t_n^{\#2}$	0	0	1	4	19	104	614	3816	24595	162896	1101922	7580904
$t_n^{\#3}$	0	0	1	2	9	46	262	1588	10053	65686	439658	2999116

Table 4: The below truth tables, (merged into one), for the five bracketed implications in $n = 4$ variables. Where Case 1 is in red, case 2 in black, case 3 in green and case 4 is indicated in blue.

p_1	p_2	p_3	p_4	$p_1 \to (p_2 \to (p_3 \to p_4))$	$p_1 \to ((p_2 \to p_3) \to p_4)$	$(p_1 \to (p_2 \to p_3)) \to p_4$	$((p_1 \to p_2) \to p_3) \to p_4$	$(p_1 \to p_2) \to (p_3 \to p_4)$
1	1	1	1	**1**	**1**	**1**	**1**	**1**
1	1	1	0	**0**	**0**	**0**	**0**	**0**
1	1	0	1	**1**	**1**	**1**	**1**	**1**
1	1	0	0	**1**	**1**	**1**	**1**	**1**
1	0	1	1	**1**	**1**	**1**	**1**	**1**
1	0	1	0	**1**	**0**	**0**	**0**	**1**
1	0	0	1	**1**	**1**	**1**	**1**	**1**
1	0	0	0	**1**	**0**	**0**	**0**	**1**
0	1	1	1	**1**	**1**	**1**	**1**	**1**
0	1	1	0	**1**	**1**	**0**	**0**	**0**
0	1	0	1	**1**	**1**	**1**	**1**	**1**
0	1	0	0	**1**	**1**	**0**	**1**	**1**
0	0	1	1	**1**	**1**	**1**	**1**	**1**
0	0	1	0	**1**	**1**	**0**	**0**	**0**
0	0	0	1	**1**	**1**	**1**	**1**	**1**
0	0	0	0	**1**	**1**	**1**	**1**	**1**

5 Asymptotic Estimate 1

In this chapter we will mainly make use of the asymptotic techniques that we have mentioned in [7, pg 6-7], and in [2, pg 6-7]. In [2] we have shown that the following asymptotic results for the sequence $\{f_n\}_{n\geq 1}$ are true:

Theorem 5.1 *Let f_n be number of rows with the value false in the truth tables of all the bracketed implications with n variables. Then*

$$\boxed{f_n \sim \left(\frac{3-\sqrt{3}}{6}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}}.}$$

Corollary 5.2 *Let g_n be the total number of rows in all truth tables for bracketed implications with n variables, and f_n the number of rows with the value "false". Then $\lim_{n\to\infty} f_n/g_n = (3-\sqrt{3})/6$.*

Theorem 5.3 *Let $t_n^{\#3}$ be number of rows with the value true in the truth tables of all the bracketed implications with n variables, case 3. Then*

$$\boxed{t_n^{\#3} \sim \left(\frac{2\sqrt{3}-3}{6}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}}}$$

Proof Recall that

$$T_3(x) = \frac{2+2\sqrt{1-8x}-\sqrt{2+2\sqrt{1-8x}+8x}-\sqrt{1-8x}\sqrt{2+2\sqrt{1-8x}+8x}}{8}.$$

By using the asymptotic techniques which we have discussed in [2] and [7], we found that $r=\frac{1}{8}$, $f(x)=\sqrt{1-8x}$ and $T_3(\frac{1}{8})=\frac{2-\sqrt{3}}{8}\neq 0$. Let $A(x)=T_3(x)-T_3(\frac{1}{8})$

$$\lim_{x\to 1/8}\frac{A(x)}{f(x)} = \lim_{x\to 1/8}\frac{2\sqrt{1-8x}-\sqrt{2+2\sqrt{1-8x}+8x}-\sqrt{1-8x}\sqrt{2+2\sqrt{1-8x}+8x}+\sqrt{3}}{8\sqrt{1-8x}}.$$

Let $v=\sqrt{1-8x}$. Then

$$\begin{aligned}
L &= \lim_{v\to 0}\frac{2v-\sqrt{(1+v)(3-v)}-v\sqrt{(1+v)(3-v)}+\sqrt{3}}{8v}\\
&= \lim_{v\to 0}\frac{2v-\sqrt{3+2v-v^2}-v\sqrt{3+2v-v^2}+\sqrt{3}}{8v}\\
&= \lim_{v\to 0}\frac{2(\sqrt{-(v+1)(v-3)}-2-v+v^2)}{8\sqrt{-(v+1)(v-3)}}\\
&= -\frac{2\sqrt{3}-3}{12},
\end{aligned}$$

where we have used l'Hôpital's Rule in the penultimate line.

Finally,

$$t_n^{\#3} \sim -\frac{2\sqrt{3}-3}{12}\binom{n-\frac{3}{2}}{n}\left(\frac{1}{8}\right)^{-n} \sim \left(\frac{2\sqrt{3}-3}{6}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}},$$

and the proof is finished. $\star$

Recall that

$$T_2(x) = F(x) - x = \frac{-1 - \sqrt{1-8x} + \sqrt{2 + 2\sqrt{1-8x} + 8x} - 4x}{4}.$$

Theorem 5.4 *Let $t_n^{\#2}$ be number of rows with the value true in the truth tables of all the bracketed implications with n variables, case 2. Then*

$$\boxed{t_n^{\#2} \sim \left(\frac{3-\sqrt{3}}{6}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}.}$$

Proof Same as for f_n, see [2] . $\star$

Theorem 5.5 *Let $t_n^{\#1}$ be number of rows with the value true in the truth tables of all the bracketed implications with n variables, case 1. Then*

$$\boxed{t_n^{\#1} \sim \left(\frac{1}{2}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}}$$

Proof Recall that

$$T_1(x) = \frac{6 - 2\sqrt{1-8x} - 3\sqrt{2+2\sqrt{1-8x}+8x} + \sqrt{1-8x}\sqrt{2+2\sqrt{1-8x}+8x}}{8}$$

By using the asymptotic techniques which we have discussed in [2] and [7], we found that $r = \frac{1}{8}$, $f(x) = \sqrt{1-8x}$ and $T_1(\frac{1}{8}) = \frac{6-3\sqrt{3}}{8} \neq 0$. Let $A(x) = T_1(x) - T_1(\frac{1}{8})$, then

$$\lim_{x\to 1/8} \frac{A(x)}{f(x)} = \lim_{x\to 1/8} \frac{-2\sqrt{1-8x} - 3\sqrt{2+2\sqrt{1-8x}+8x} + \sqrt{1-8x}\sqrt{2+2\sqrt{1-8x}+8x} + 3\sqrt{3}}{8\sqrt{1-8x}}.$$

Let $v = \sqrt{1-8x}$. Then

$$\begin{aligned} L &= \lim_{v\to 0} \frac{-2v - 3\sqrt{(1+v)(3-v)} + v\sqrt{(1+v)(3-v)} + 3\sqrt{3}}{8v} \\ &= \lim_{v\to 0} \frac{-2v - 3\sqrt{3+2v-v^2} + v\sqrt{3+2v-v^2} + \sqrt{3}}{8v} \\ &= \lim_{v\to 0} -\frac{2(\sqrt{-(v+1)(v-3)} - 3v + v^2)}{8\sqrt{-(v+1)(v-3)}} \\ &= -\frac{1}{4}, \end{aligned}$$

where we have used l'Hôpital's Rule in the penultimate line. Finally,

$$t_n^{\#1} \sim -\frac{1}{4}\binom{n-\frac{3}{2}}{n}\left(\frac{1}{8}\right)^{-n} \sim \left(\frac{1}{2}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}},$$

and the proof is finished. $\star$

The importance of the constants $\frac{2\sqrt{3}-3}{6} = 0.077350269189$, $\frac{3-\sqrt{3}}{6} = 0.211324865404$ and $\frac{1}{2} = 0.5$ lies in the following fact:

Corollary 5.6 *Let g_n be the total number of rows in all truth tables for bracketed implications with n variables, then*

$$\lim_{n\to\infty} \frac{t_n^{\#3}}{g_n} = \frac{2\sqrt{3}-3}{6}, \quad \lim_{n\to\infty} \frac{f_n}{g_n} = \lim_{n\to\infty} \frac{t_n^{\#2}}{g_n} = \frac{3-\sqrt{3}}{6}, \quad \lim_{n\to\infty} \frac{t_n^{\#1}}{g_n} = \frac{1}{2}.$$

The table below illustrates the convergence :

n	$f_n = t_n^{\#2}$	g_n	$t_n^{\#2}/g_n$	$t_n^{\#1}$	$t_n^{\#1}/g_n$	$t_n^{\#3}$	$t_n^{\#3}/g_n$
1	1 –	2	0.5 –	–	–	–	–
2	2	4	0.25	1	0.25	1	0.25
3	4	16	0.25	6	0.375	2	0.125
4	19	80	0.2375	33	0.4125	9	0.1125
5	104	428	0.2321428571	194	0.433035714	46	0.102678571
6	614	2688	0.228422619	1198	0.445684524	262	0.0974702381
7	3816	16896	0.2258522727	7676	0.454308712	1588	0.0939867424
8	424595	109824	0.2239492279	50581	0.460564175	10053	0.0915373689
9	162896	732160	0.2224868881	340682	0.465310861	65686	0.0897153628
10	1101922	4978688	0.2213277876	2335186	0.469036421	439658	0.0883080040
100	–	–	0.212290865	–	0.497093847	–	0.0783244229

Corollary 5.7 *Let $\mathcal{P}^{\#1} = \frac{t_n^{\#1}}{g_n}$, $\mathcal{P}^{\#2} = \frac{t_n^{\#2}}{g_n}$, $\mathcal{P}^{\#3} = \frac{t_n^{\#3}}{g_n}$, and $\mathcal{P}^{\#4} = \frac{f_n}{g_n}$, then for $n > 2$,*

$$\mathcal{P}^{\#1} > (\mathcal{P}^{\#2} = \mathcal{P}^{\#4}) > \mathcal{P}^{\#3}.$$

Corollary 5.8

$$\lim_{n\to\infty} \frac{t_n^{\#3}}{t_n^{\#2}} = \frac{\sqrt{3}-1}{2}, \quad \lim_{n\to\infty} \frac{t_n^{\#2}}{t_n^{\#3}} = 1+\sqrt{3}, \quad \lim_{n\to\infty} \frac{t_n^{\#3}}{t_n^{\#1}} = \frac{2\sqrt{3}-3}{3},$$

$$\lim_{n\to\infty} \frac{t_n^{\#1}}{t_n^{\#3}} = 3+2\sqrt{3}, \quad \lim_{n\to\infty} \frac{t_n^{\#2}}{t_n^{\#1}} = \frac{3-\sqrt{3}}{3}, \quad \lim_{n\to\infty} \frac{t_n^{\#1}}{t_n^{\#2}} = \frac{3+\sqrt{3}}{2}.$$

6 Counting true entries in truth tables of bracketed formulae connected by modified implication

A number of new enumerative problems arise if we modify the binary connective of implication as in below.

6.1 Type 1

Definition 6.1 *Let ψ, and ϕ be propositional variables then*

$$\psi \rightharpoonup \phi \equiv \psi \to \neg\phi.$$

Thus for any valuation ν,

$$\nu(\psi \rightharpoonup \phi) = \begin{cases} 0 & \textit{if } \nu(\psi) = 1 \textit{ and } \nu(\phi) = 1, \\ 1 & \textit{otherwise;} \end{cases}$$

$$\nu(\psi \rightharpoonup \phi) = 1 \quad : \quad (\nu(\psi) = 0 \vee \nu(\phi) = 0)$$

Therefore there are three cases to consider here:

$$\underbrace{(\nu(\psi) = 0 \wedge \nu(\phi) = 0)}_{\textit{case 1}} \vee \underbrace{(\nu(\psi) = 0 \wedge \nu(\phi) = 1)}_{\textit{case 2}} \vee \underbrace{(\nu(\psi) = 1 \wedge \nu(\phi) = 0)}_{\textit{case 3}}$$

Addition to the three cases in above there is also the fourth case, known as the *disastrous combination* :

$$\nu(\psi \rightharpoonup \phi) = 0 \Longleftrightarrow \underbrace{(\nu(\psi) = 1 \wedge \nu(\phi) = 1)}_{\textit{case 4}}.$$

Let y_n, and d_n be the number of rows with the value "false" and "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, type 1, respectively. Then using theorem 3.1,

$$g_n = \sum_{i=1}^{n-1} (y_i + d_i)(y_{n-i} + d_{n-i})$$

Thus if we expand the right hand side

$$g_n = \underbrace{\sum_{i=1}^{n-1} y_i y_{n-i}}_{\textit{case 1}} + \underbrace{\sum_{i=1}^{n-1} y_i d_{n-i}}_{\textit{case 2}} + \underbrace{\sum_{i=1}^{n-1} d_i y_{n-i}}_{\textit{case 3}} + \underbrace{\sum_{i=1}^{n-1} d_i d_{n-i}}_{\textit{case 4}} \tag{8}$$

The four cases in equation (8) coincide with the four cases in the penultimate lines respectively. Let

$$d_n^{\#1} = \sum_{i=1}^{n-1} y_i y_{n-i}, \quad d_n^{\#2} = \sum_{i=1}^{n-1} y_i d_{n-i}, \quad d_n^{\#3} = \sum_{i=1}^{n-1} d_i y_{n-i}, \quad y_n = \sum_{i=1}^{n-1} d_i d_{n-i}.$$

Where

$$0 = d_1^{\#1} = d_1^{\#2} = d_1^{\#3}, \quad \textit{and} \quad y_1 = 1.$$

6.1.1 Case 4

This case has been manifested in [7], and in summary we had the following results:

Theorem 6.2 *Let y_n be the number of rows with the value "false" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, type 1, case 4. Then*

$$\boxed{y_n = \sum_{i=1}^{n-1} \left((2^i C_i - y_i)(2^{n-i} C_{n-i} - y_{n-i}) \right), \ \textit{with } y_0 = 0, \ y_1 = 1. \quad (9)}$$

and for large n, $y_n \sim \left(\frac{10 - 2\sqrt{10}}{20} \right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}$. Where C_i is the ith Catalan number.

If we look closely to the recurrence relation (9), since $d_i = (2^i C_i - y_i)$ we obtain the 'case 4' in equation (8):

$$y_n = \sum_{i=1}^{n-1} d_i d_{n-i}, \quad \textit{with} \ \ y_1 = 1.$$

The first ten terms of the sequence $\{y_n\}_{n \geq 1}$ are:

$$1, 1, 6, 29, 162, 978, 6156, 40061, 267338, 819238, \ldots$$

We have also shown in [7] that y_n has the following generating function

$$\boxed{Y(x) = \frac{2 - \sqrt{1 - 8x} - \sqrt{3 - 4x - 2\sqrt{1 - 8x}}}{2}.}$$

We have also shown in [7] that the sequence $\{y_n\}_{n \geq 1}$ preserves the parity of Catalan numbers.

Table 5: $n = 3$

p_1	p_2	p_3	$p_1 \rightharpoonup (p_2 \rightharpoonup p_3)$	$(p_1 \rightharpoonup p_2) \rightharpoonup p_3$
1	1	1	1	1
1	1	0	**0**	**1**
1	0	1	**0**	**0**
1	0	0	**0**	**1**
0	1	1	**1**	**0**
0	1	0	**1**	**1**
0	0	1	**1**	**0**
0	0	0	**1**	**1**

6.1.2 Case 3

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \rightharpoonup \phi) = 1 \quad : \quad (\nu(\psi) = 1 \ \wedge \ \nu(\phi) = 0).$$

In this case we are interested in formulae obtained from $p_1 \rightharpoonup \ldots \rightharpoonup p_n$ by inserting brackets so that the valuation of the first i bracketing give 1 and the rest $(n-i)$ bracketing give 0. The table 5 below shows the truth tables, (merged into one), for the two bracketed implications in $n = 3$ variables; where the corresponding case 3 truth values are denoted in green.

Proposition 6.3 *Let $d_n^{\#3}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, type 1, such that the valuation of the first i bracketing gives 1 and the rest of $(n-i)$ bracketing gives 0. Then,*

$$d_n^{\#3} = \sum_{i=1}^{n-1} d_i y_{n-i}, \ \textit{where } \{y_i\} \textit{ is defined in equation (9)} . \tag{10}$$

Since $d_i = (2^i C_i - y_i)$,

$$d_n^{\#3} = \sum_{i=1}^{n-1} (2^i C_i - y_i) y_{n-i}.$$

Proof A row with the value 1, 'true', comes from an expression $\psi \rightharpoonup \phi$, where $\nu(\psi) = 1$ and $\nu(\phi) = 0$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Let $D_3(x) = \sum_{n \geq 1} d_n^{\#3} x^n$, then $D_3(x) = \sum_{n \geq 1} \sum_{i=1}^{n-1} (2^i C_i - y_i) y_{n-i} x^n$, which gives us $D_3(x) = (G(x) - Y(x))Y(x)$, writing this more explicitly gives us the following proposition:

Proposition 6.4 *The generating function for the sequence $\{d_n^{\#3}\}_{n \geq 0}$ is given by*

$$\boxed{D_3(x) = \frac{-5 + 3\sqrt{1-8x} + 3\sqrt{3 - 4x - 2\sqrt{1-8x}} + 4x - \sqrt{1-8x}\sqrt{3 - 4x - 2\sqrt{1-8x}}}{4}.}$$

By using Maple we find the first 25 terms of this sequence:

$$\begin{aligned} \{d_n^{\#3}\}_{n \geq 2} \ = \ & 1, 4, 19, 108, 646, 4056, 26355, 175628, 1193906, 8246856, \\ & 57716798, 408391736, 13 + 2916689516, 20997741104, 152218453443, \\ & 1110202813836, 8140864778810, 59981252880360, 443834410644618, \\ & 3296876425605992, 24575508928455572, 183773880824034512, \\ & 1378248141659861486, 10364040821146016568 \ldots \end{aligned}$$

6.1.3 Case 2

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \rightharpoonup \phi) = 1 \quad : \quad (\nu(\psi) = 0 \ \wedge \ \nu(\phi) = 1).$$

In this case we are interested in formulae obtained from $p_1 \rightharpoonup \ldots \rightharpoonup p_n$ by inserting brackets so that the valuation of the first i bracketing gives 0 and the rest $(n-i)$ bracketing give 1. The table 5 above shows the truth tables, (merged into one), for the two bracketed implications in $n = 3$ variables; where the corresponding case 3 truth values are denoted in red.

Proposition 6.5 *Let $d_n^{\#2}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, type 1, such that the valuation of the first i bracketing gives 0 and the rest of $(n-i)$ bracketing gives 1. Then,*

$$d_n^{\#2} = \sum_{i=1}^{n-1} y_i d_{n-i}, \text{ where } \{y_i\} \text{ is defined in equation (9).} \tag{11}$$

Since $d_i = (2^i C_i - y_i)$,

$$d_n^{\#2} = \sum_{i=1}^{n-1} y_i (2^{n-i} C_{n-i} - y_{n-i}).$$

Proof A row with the value 1, 'true', comes from an expression $\psi \rightharpoonup \phi$, where $\nu(\psi) = 0$ and $\nu(\phi) = 1$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Let $D_2(x) = \sum_{n\geq 1} d_n^{\#2} x^n$, then $D_2(x) = \sum_{n\geq 1} \sum_{i=1}^{n-1} y_i (2^{n-i} C_{n-i} - y_{n-i}) x^n$, which gives us $D_2(x) = Y(x)(G(x) - Y(x))$, thus Case 2 coincides with Case 3. Writing this more explicitly gives us the following proposition:

Proposition 6.6 *The generating function for the sequence $\{d_n^{\#2}\}_{n\geq 0}$ is given by*

$$\boxed{D_2(x) = \frac{-5 + 3\sqrt{1-8x} + 3\sqrt{3 - 4x - 2\sqrt{1-8x}} + 4x - \sqrt{1-8x}\sqrt{3 - 4x - 2\sqrt{1-8x}}}{4}.}$$

6.1.4 Case 1

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \rightharpoonup \phi) = 1 \quad : \quad (\nu(\psi) = 0 \ \wedge \ \nu(\phi) = 0).$$

In this case we are interested in formulae obtained from $p_1 \rightharpoonup \ldots \rightharpoonup p_n$ by inserting brackets so that the valuation of the first i bracketing gives 0 and the rest $(n-i)$ bracketing gives 0. The table 5 above shows the truth tables, (merged into one), for the two bracketed implications in $n = 3$ variables; where the corresponding case 3 truth values are denoted in black.

Proposition 6.7 *Let $d_n^{\#1}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, type 1, such that the valuation of the first i bracketing gives 0 and the rest of $(n-i)$ bracketing gives 0. Then,*

$$d_n^{\#1} = \sum_{i=1}^{n-1} y_i y_{n-i}, \text{ where } \{y_i\} \text{ is defined in equation (9).} \tag{12}$$

Proof A row with the value 1, 'true', comes from an expression $\psi \rightharpoonup \phi$, where $\nu(\psi) = 0$ and $\nu(\phi) = 0$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Let $D_1(x) = \sum_{n\geq 1} d_n^{\#1} x^n$, then $D_1(x) = \sum_{n\geq 1}\sum_{i=1}^{n-1} y_i y_{n-i} x^n$, which gives us $D_2(x) = Y(x)^2$. Writing this more explicitly gives us the following proposition:

Proposition 6.8 *The generating function for the sequence $\{d_n^{\#1}\}_{n\geq 0}$ is given by*

$$\boxed{D_1(x) = \frac{4 - 3\sqrt{1-8x} - 2\sqrt{3-4x-2\sqrt{1-8x}} - 6x + \sqrt{1-8x}\sqrt{3-4x-2\sqrt{1-8x}}}{2}.}$$

By using Maple we find the first 25 terms of this sequence:

$$\begin{aligned}\{d_n^{\#1}\}_{n\geq 2} \;=\;& 1, 2, 13, 70, 418, 2628, 17053, 113566, 771638, 5327804, 37274482, 263669500,\\ & 1882630692, 13550468360, 98212733277, 716195167502, 5250931034798,\\ & 8683418448780, 286206574421222, 2125766544922612, 15844332066531484,\\ & 3118472460044221368, 888436633672089842, 6680306733514013388, \ldots\end{aligned}$$

6.2 Type 2

Definition 6.9 *Let ψ, and ϕ be propositional variables then*

$$\psi \leftharpoonup \phi \equiv \neg\psi \to \phi.$$

For any valuation ν,

$$\nu(\psi \rightharpoonup \phi) = \begin{cases} 0 & \text{if } \nu(\psi) = 0 \text{ and } \nu(\phi) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Let k_n, h_n be the number of rows with the value "true", and "false" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, in the type (ii), respectively. Then using theorem 3.1,

$$g_n = \sum_{i=1}^{n-1}(h_i + k_i)(h_{n-1} + k_{n-i})$$

Thus if we expand the right hand side

$$g_n = \underbrace{\sum_{i=1}^{n-1} h_i h_{n-i}}_{\textit{case 4}} + \underbrace{\sum_{i=1}^{n-1} h_i k_{n-i}}_{\textit{case 3}} + \underbrace{\sum_{i=1}^{n-1} k_i h_{n-i}}_{\textit{case 2}} + \underbrace{\sum_{i=1}^{n-1} k_i k_{n-i}}_{\textit{case 1}}. \tag{13}$$

Let

$$k_n^{\#1} = \sum_{i=1}^{n-1} k_i k_{n-i}, \quad k_n^{\#2} = \sum_{i=1}^{n-1} k_i h_{n-i}, \quad k_n^{\#3} = \sum_{i=1}^{n-1} h_i k_{n-i}, \quad h_n = \sum_{i=1}^{n-1} h_i h_{n-i}.$$

Where

$$0 = k_1^{\#1} = k_1^{\#2} = k_1^{\#3}, \quad \textit{and} \quad h_1 = 1.$$

6.2.1 Case 4

This case has already been discussed in [7], and we had the following results:

Proposition 6.10 *Let h_n be the number of rows with the value "false" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, type 2, such that the valuation of the first i bracketing gives 0 and the rest of $(n-i)$ bracketing gives 0. Then,*

$$h_n = \sum_{i=1}^{n-1} h_i h_{n-i}, \text{ where } h_0 = 0, h_1 = 1. \tag{14}$$

The recurrence relation (14) is very well known; it is the recurrence relation for Catalan numbers. The Catalan numbers has the following generating function, and the explicit formula:

$$H(x) = \frac{1-\sqrt{1-4x}}{2}, \quad h_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Corollary 6.11 *Suppose we have all possible well-formed formulae obtained from $p_1 \leftharpoonup p_2 \leftharpoonup \ldots \leftharpoonup p_n$ by inserting brackets, where $p_1, \ldots, p_n$ are distinct propositions. Then each formula defines the same truth table, (provided that we do not distinguish the 1s by their case type).*

Here are the truth tables, (merged into one), for the bracketed m-implications, in $n = 3$ variables. Where the corresponding false entries are denoted in blue:

Table 6: $n = 3$

p_1	p_2	p_3	$p_1 \leftharpoonup (p_2 \leftharpoonup p_3)$	$(p_1 \leftharpoonup p_2) \leftharpoonup p_3$
1	1	1	**1**	**1**
1	1	0	**1**	**1**
1	0	1	**1**	**1**
1	0	0	**1**	**1**
0	1	1	**1**	**1**
0	1	0	**1**	**1**
0	0	1	**1**	**1**
0	0	0	**0**	**0**

6.2.2 Case 3

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \leftharpoonup \phi) = 1 \quad : \quad (\nu(\psi) = 0 \wedge \nu(\phi) = 1).$$

In this case we are interested in formulae obtained from $p_1 \leftharpoonup p_2 \leftharpoonup \ldots \leftharpoonup p_n$ by inserting brackets so that the valuation of the first i bracketing gives 0 and the rest $(n-i)$ bracketing gives 0. The table 6 above shows the truth tables, (merged into one), for the two bracketed implications in $n = 3$ variables. The corresponding case 3 truth values are denoted in red.

Proposition 6.12 *Let $k_n^{\#3}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, type 2, such that the valuation of the first i bracketing gives 0 and the rest of $(n-i)$ bracketing gives 1. Then,*

$$k_n^{\#3} = \sum_{i=1}^{n-1} h_i k_{n-i}, \text{ where } \{h_i\} \text{ is defined in equation (14).} \tag{15}$$

Proof A row with the value 1, 'true', comes from an expression $\psi \rightharpoonup \phi$, where $\nu(\psi) = 0$ and $\nu(\phi) = 1$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Since $k_i = (2^i C_i - h_i)$, from equation (15) we get,

$$k_n^{\#3} = \sum_{i=1}^{n-1} h_i(2^{n-i}C_{n-i} - h_{n-i}).$$

Let $K_3(x) = \sum_{i\geq 1} k_n^{\#3} x^n$, then we get $K_3(x) = H(x)(G(x) - H(x))$, writing the right hand side more explicitly gives the following proposition:

Proposition 6.13 *The generating function for the sequence $\{k_n^{\#3}\}_{n\geq 0}$ is given by*

$$\boxed{K_3(x) = \frac{-1 - \sqrt{1-8x} + \sqrt{1-8x}\sqrt{1-4x} + \sqrt{1-4x} + 4x}{4}.}$$

By using Maple we find the first 25 terms of this sequence:

$$\begin{aligned}\{k_n^{\#3}\}_{n\geq 2} \quad = \quad & 1, 4, 19, 100, 566, 3384, 21107, 136084, 900674, 6087496, \\ & 41850366, 291766952, 2057964492, 14659421040, 105305580483, 761981900724, \\ & 5548736343434, 0632122219688, 299017702596554, 2210275626304248, \\ & 16403005547059508, 122169144755555088, 912887876722311406, 684174390763667239, \ldots\end{aligned}$$

6.2.3 Case 2

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \leftharpoonup \phi) = 1 \quad : \quad (\nu(\psi) = 1 \ \wedge \ \nu(\phi) = 0).$$

In this case we are interested in formulae obtained from $p_1 \leftharpoonup p_2 \leftharpoonup \ldots \leftharpoonup p_n$ by inserting brackets so that the valuation of the first i bracketing gives 1 and the rest $(n-i)$ bracketing gives 0. The table 6 above shows the truth tables, (merged into one), for the two bracketed implications in $n = 3$ variables. The corresponding case 3 truth values are denoted in green.

Proposition 6.14 *Let $k_n^{\#2}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, type 2, such that the valuation of the first i bracketing gives 1 and the rest of $(n-i)$ bracketing gives 0. Then,*

$$k_n^{\#2} = \sum_{i=1}^{n-1} k_i h_{n-i}, \ \textit{where } \{h_i\} \textit{ is defined in equation (14).} \tag{16}$$

Proof A row with the value 1, 'true', comes from an expression $\psi \rightharpoonup \phi$, where $\nu(\psi) = 1$ and $\nu(\phi) = 0$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Since $k_i = (2^i C_i - h_i)$, from equation (16) we get,

$$k_n^{\#2} = \sum_{i=1}^{n-1} (2^i C_i - h_i) h_{n-i}.$$

Let $K_2(x) = \sum_{i\geq 1} k_n^{\#2} x^n$, then we get $K_2(x) = (G(x) - H(x))H(x)$, writing the right hand side more explicitly gives the following proposition:

Proposition 6.15 *The generating function for the sequence $\{k_n^{\#2}\}_{n\geq 0}$ is given by*

$$\boxed{K_2(x) = \frac{-1-\sqrt{1-8x}+\sqrt{1-8x}\sqrt{1-4x}+\sqrt{1-4x}+4x}{4}.}$$

6.2.4 Case 1

Let ψ and ϕ be propositional variables, then

$$\nu(\psi \leftharpoonup \phi) = 1 \quad : \quad (\nu(\psi) = 1 \ \wedge \ \nu(\phi) = 1).$$

In this case we are interested in formulae obtained from $p_1 \leftharpoonup p_2 \leftharpoonup \ldots \leftharpoonup p_n$ by inserting brackets so that the valuation of the first i bracketing gives 1 and the rest $(n-i)$ bracketing gives 1. The table 6 above shows the truth tables, (merged into one), for the two bracketed implications in $n=3$ variables. The corresponding case 3 truth values are denoted in black.

Proposition 6.16 *Let $k_n^{\#1}$ be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct propositions $p_1,\ldots,p_n$ connected by the binary connective of m-implication, type 2, such that the valuation of the first i bracketing gives 1 and the rest of $(n-i)$ bracketing gives 1. Then,*

$$k_n^{\#1} = \sum_{i=1}^{n-1} k_i k_{n-i}. \tag{17}$$

Proof A row with the value 1, 'true', comes from an expression $\psi \rightharpoonup \phi$, where $\nu(\psi) = 1$ and $\nu(\phi) = 1$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Since $k_i = (2^i C_i - h_i)$, from equation (15) we get,

$$k_n^{\#2} = \sum_{i=1}^{n-1} (2^i C_i - h_i)(2^{n-i} C_{n-i} - h_{n-i}), \qquad \textit{where } h_i \textit{ is the sequence in (14).}$$

Let $K_1(x) = \sum_{i\geq 1} k_n^{\#1} x^n$, then we get $K_2(x) = (G(x) - H(x))^2$, writing the right hand side more explicitly gives the following proposition:

Proposition 6.17 *The generating function for the sequence $\{k_n^{\#1}\}_{n\geq 0}$ is given by*

$$\boxed{K_1(x) = \frac{1-6x-\sqrt{1-4x}\sqrt{1-8x}}{2}.}$$

By using Maple we find the first 25 terms of this sequence:

$$\begin{aligned}\{k_n^{\#1}\}_{n\geq 2} \;=\;& 1, 6, 37, 234, 514, 9996, 67181, 458562, 3172478, 22206420,\\ & 157027938, 1120292388, 8055001716, 58314533400, 424740506109, 3110401363122,\\ & 22888001498102, 169155516667524, 1255072594261142, 9345400450314924,\\ & 69812926066668044, 523072984217339304, 3929809142578361938, 29598511892723647860, \ldots\end{aligned}$$

6.3 Type 3

Definition 6.18 *Let ψ, and ϕ be propositional variables then*

$$\psi \rightleftharpoons \phi \equiv \neg\psi \to \neg\phi.$$

For any valuation ν,

$$\nu(\psi \rightleftharpoons \phi) = \begin{cases} 0 & \text{if } \nu(\psi) = 0 \text{ and } \nu(\phi) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Let b_n, s_n be the number of rows with the value "true", and "false" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \dots, p_n$ connected by the binary connective of m-implication, in the type (iii), respectively. Then using theorem 3.1,

$$g_n = \sum_{i=1}^{n-1} (b_i + s_i)(b_{n-i} + s_{n-i})$$

Thus if we expand the right hand side

$$g_n = \underbrace{\sum_{i=1}^{n-1} b_i b_{n-i}}_{\text{case 1}} + \underbrace{\sum_{i=1}^{n-1} b_i s_{n-i}}_{\text{case 2}} + \underbrace{\sum_{i=1}^{n-1} s_i b_{n-i}}_{\text{case 4}} + \underbrace{\sum_{i=1}^{n-1} s_i s_{n-i}}_{\text{case 3}}. \tag{18}$$

Let

$$b_n^{\#1} = \sum_{i=1}^{n-1} b_i b_{n-i}, \quad b_n^{\#2} = \sum_{i=1}^{n-1} b_i s_{n-i}, \quad b_n^{\#3} = \sum_{i=1}^{n-1} s_i s_{n-i}, \quad s_n = \sum_{i=1}^{n-1} s_i b_{n-i}.$$

Where

$$0 = b_1^{\#1} = b_1^{\#2} = b_1^{\#3}, \quad \text{and} \quad s_1 = 1.$$

Proposition 6.19 *Let $s_n^{\#3}$ be the number of rows with the value "false" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \dots, p_n$ connected by the binary connective of m-implication, type 3, such that the valuation of the first i bracketing gives 0 and the rest of $(n-i)$ bracketing gives 1. Then,*

$$s_n = \sum_{i=1}^{n-1} s_i b_{n-i}, \text{ where } s_0 = 0, s_1 = 1. \tag{19}$$

Since $b_i = 2^i C_i - s_i$,

$$s_n = \sum_{i=1}^{n-1} s_i (2^{n-i} C_{n-i} - s_{n-i}),$$

Proof A row with the value 1, 'true', comes from an expression $\psi \rightleftharpoons \phi$, where $\nu(\psi) = 0$ and $\nu(\phi) = 1$. If ψ contains i variables, then ϕ contains $(n-i)$ variables, and the number of choices is given by the summand in the proposition. $\star$

Since the recurrence relation (19), is equivalent to the recurrence relation for the sequence $\{f_n\}_{n\geq 1}$, the four generating function in section 3, (starting from page 5), and the corresponding asymptotics, (starting from page 11), will appear again. Thus the reader is suggested to work these cases on his-own.

7 Asymptotic Estimate 2

7.1 Asymptotic for Type 1 and 3

In [7] we have shown that the following asymptotic results are true for the sequence $\{y_n\}_{n\geq 1}$:

Theorem 7.1 *Let y_n be number of rows with the value false in the truth tables of all the bracketed m-implications, case(i), with n distinct variables. Then*

$$\boxed{y_n \sim \left(\frac{10-2\sqrt{10}}{10}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}}.}$$

Corollary 7.2 *Let g_n be the total number of rows in all truth tables for bracketed m-implications, case(i), with n distinct variables, and y_n the number of rows with the value "false". Then* $\lim_{n\to\infty} y_n/g_n = \frac{10-2\sqrt{10}}{10}$.

Theorem 7.3 *Let $d_n^{\#3}$ be number of rows with the value true in the truth tables of all the bracketed implications with n variables, type 1, case 3. Then*

$$\boxed{d_n^{\#3} \sim \left(\frac{11\sqrt{10}-30}{20}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}}.}$$

Proof Recall that

$$D_3(x) = \frac{-5+3\sqrt{1-8x}+3\sqrt{3-4x-2\sqrt{1-8x}}+4x-\sqrt{1-8x}\sqrt{3-4x-2\sqrt{1-8x}}}{4}.$$

By using the asymptotic techniques which we have discussed in [2] and [7], we found that $r=\frac{1}{8}$, $f(x)=\sqrt{1-8x}$ and $D_3(\frac{1}{8}) = \frac{-9+3\sqrt{10}}{8} \neq 0$. Let $A(x) = D_3(x) - D_3(\frac{1}{8})$

$$\lim_{x\to 1/8}\frac{A(x)}{f(x)} = \lim_{x\to 1/8}\frac{-1+6\sqrt{1-8x}+6\sqrt{2-4x-2\sqrt{1-8x}}-2\sqrt{1-8x}\sqrt{2-4x-2\sqrt{1-8x}}+8x-3\sqrt{10}}{8\sqrt{1-8x}}.$$

Let $v=\sqrt{1-8x}$. Then

$$\begin{aligned}
L &= \lim_{v\to 0}\frac{6v+3\sqrt{2}\sqrt{5-4v+v^2}-\sqrt{2}v\sqrt{5-4v+v^2}-v^2-3\sqrt{10}}{8v}\\
&= \lim_{v\to 0} -\frac{-6\sqrt{5-4v+v^2}+11\sqrt{2}-9\sqrt{2}v^2+2v\sqrt{5-4v-v^2}}{8\sqrt{5-4v-v^2}}\\
&= -\frac{\sqrt{5}(-6\sqrt{5}+11\sqrt{2})}{40}\\
&= -\frac{11\sqrt{10}-30}{40},
\end{aligned}$$

where we have used l'Hôpital's Rule in the penultimate line.

Finally,

$$t_n^{\#3} \sim -\frac{11\sqrt{10}-30}{40}\binom{n-\frac{3}{2}}{n}\left(\frac{1}{8}\right)^{-n} \sim \left(\frac{11\sqrt{10}-30}{20}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}},$$

and the proof is finished. ⋆

Theorem 7.4 *Let $d_n^{\#2}$ be number of rows with the value true in the truth tables of all the bracketed implications with n variables, type 1, case 2. Then*

$$\boxed{d_n^{\#2} \sim \left(\frac{11\sqrt{10}-30}{20}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}}.}$$

Theorem 7.5 *Let $d_n^{\#1}$ be number of rows with the value true in the truth tables of all the bracketed implications with n variables, type 1, case 1. Then*

$$\boxed{d_n^{\#1} \sim \left(\frac{20-9\sqrt{10}}{10}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}}.}$$

Proof Recall that

$$D_1(x) = \frac{4-3\sqrt{1-8x}-2\sqrt{3-4x-2\sqrt{1-8x}}-6x+\sqrt{1-8x}\sqrt{3-4x-2\sqrt{1-8x}}}{2}.$$

By using the asymptotic techniques which we have discussed in [2] and [7], we found that $r=\frac{1}{8}$, $f(x)=\sqrt{1-8x}$ and $D_1(\frac{1}{8})=\frac{13-4\sqrt{10}}{8}\neq 0$. Let $A(x)=D_1(x)-D_1(\frac{1}{8})$

$$\lim_{x\to 1/8}\frac{A(x)}{f(x)} = \lim_{x\to 1/8}\frac{3-12\sqrt{1-8x}-8\sqrt{2-4x-2\sqrt{1-8x}}-24x+4\sqrt{1-8x}\sqrt{2-4x-2\sqrt{1-8x}}+4\sqrt{10}}{8\sqrt{1-8x}}.$$

Let $v=\sqrt{1-8x}$. Then

$$\begin{aligned} L &= \lim_{v\to 0}\frac{-12v-4\sqrt{2}\sqrt{v^2-4v+5}+3v^2+2\sqrt{2}v\sqrt{v^2-4v+5}+4\sqrt{10}}{8v} \\ &= \lim_{v\to 0} -\frac{1}{4}\frac{-6\sqrt{v^2-4v+5}-8\sqrt{2}v+9\sqrt{2}+3v\sqrt{v^2-4v+5}+2\sqrt{2}x^2}{\sqrt{v^2-4v+5}} \\ &= -\frac{\sqrt{5}(-12\sqrt{5}+18\sqrt{2})}{40} \\ &= -\frac{30-9\sqrt{10}}{20}, \end{aligned}$$

where we have used l'Hôpital's Rule in the penultimate line.

Finally,

$$d_n^{\#1} \sim -\frac{30-9\sqrt{10}}{20}\binom{n-\frac{3}{2}}{n}\left(\frac{1}{8}\right)^{-n} \sim \left(\frac{30-9\sqrt{10}}{10}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}},$$

and the proof is finished. $\star$

Corollary 7.6 *Let g_n be the total number of rows in all truth tables for bracketed implications with n variables, then*

$$\lim_{n\to\infty}\frac{d_n^{\#1}}{g_n} = \frac{30-9\sqrt{10}}{10}, \quad \lim_{n\to\infty}\frac{d_n^{\#3}}{g_n} = \lim_{n\to\infty}\frac{d_n^{\#2}}{g_n} = \frac{11\sqrt{10}-30}{20}, \quad \lim_{n\to\infty}\frac{y_n}{g_n} = \frac{10-2\sqrt{10}}{10}.$$

Corollary 7.7

$$\lim_{n\to\infty}\frac{d_n^{\#1}}{d_n^{\#3}} = \frac{12\sqrt{10}-18}{31}, \quad \lim_{n\to\infty}\frac{d_n^{\#3}}{d_n^{\#1}} = \frac{2\sqrt{10}+3}{6}, \quad \lim_{n\to\infty}\frac{d_n^{\#1}}{y} = \frac{4-\sqrt{10}}{2},$$

$$\lim_{n\to\infty}\frac{y}{d_n^{\#1}} = \frac{4+\sqrt{10}}{3}, \quad \lim_{n\to\infty}\frac{y}{d_n^{\#3}} = \frac{16+10\sqrt{10}}{31}, \quad \lim_{n\to\infty}\frac{d_n^{\#3}}{y} = \frac{5\sqrt{10}-8}{12}.$$

7.2 Asymptotic for Type 2

In [7] we have shown that the following asymptotic results are true for the sequence $\{h_n\}_{n\geq 1}$:

Theorem 7.8 *Let h_n be number of rows with the value false in the truth tables of all the bracketed m-implications, case(i), with n distinct variables. Then*

$$\boxed{h_n \sim \frac{2^{2n}}{\sqrt{\pi n^3}}.}$$

Corollary 7.9 *Let g_n be the total number of rows in all truth tables for bracketed m-implications, case(i), with n distinct variables, and y_n the number of rows with the value "false', in type 2'. Then $\lim_{n\to\infty} h_n/g_n = 0$.*

Proof

$$\lim_{n\to\infty} \frac{h_n}{g_n} = \lim_{n\to\infty} \frac{1}{2^{n-2}} = 0.$$

$\star$

Theorem 7.10 *Let $k_n^{\#1}$ be number of rows with the value true in the truth tables of all the bracketed implications with n variables, type 1, case 1. Then*

$$\boxed{k_n^{\#3} \sim \left(\frac{\sqrt{2}}{2}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}.}$$

Proof Recall that

$$K_1(x) = \frac{1 - 6x - \sqrt{1-4x}\sqrt{1-8x}}{2}.$$

By using the asymptotic techniques which we have discussed in [2] and [7], we found that $r = \frac{1}{8}$, $f(x) = \sqrt{1-8x}$ and $K_1(\frac{1}{8}) = \frac{1}{8} \neq 0$. Let $A(x) = K_1(x) - K_1(\frac{1}{8})$

$$\lim_{x\to 1/8} \frac{A(x)}{f(x)} = \lim_{x\to 1/8} \frac{3 - 24x - 4\sqrt{1-8x}\sqrt{1-4x}}{8\sqrt{1-8x}}.$$

Let $v = \sqrt{1-8x}$. Then

$$\begin{aligned} L &= \lim_{v\to 0} \frac{3v^2 - 2\sqrt{2}v\sqrt{1+v^2}}{8v} \\ &= \lim_{v\to 0} -\frac{2(-3v\sqrt{1+v^2} + \sqrt{2} + 2\sqrt{2}v^2)}{1+\sqrt{v^2}} \\ &= -\frac{\sqrt{2}}{4}, \end{aligned}$$

where we have used l'Hôpital's Rule in the penultimate line.

Finally,

$$k_n^{\#1} \sim -\frac{\sqrt{2}}{4} \binom{n-\frac{3}{2}}{n} \left(\frac{1}{8}\right)^{-n} \sim \left(\frac{\sqrt{2}}{2}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}},$$

and the proof is finished. $\star$

Corollary 7.11 *Let g_n be the total number of rows in all truth tables for bracketed m-implications, case(i), with n distinct variables, and $k_n^{\#1}$ be the number of rows with the value 'true', in type 2. Then* $\lim_{n\to\infty} k_n^{\#1}/g_n = \frac{\sqrt{2}}{2}$.

Theorem 7.12 *Let $k_n^{\#3,2}$ be number of rows with the value true in the truth tables of all the bracketed implications with n variables, type 1, case 1. Then*

$$\boxed{k_n^{\#3,2} \sim \left(\frac{2-\sqrt{2}}{4}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}.}$$

Proof Since the total probability has to add up to 1, and $K_2(x)$ and $K_3(x)$ both have the same generating function:

$$\frac{1-\frac{\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4}.$$

$\star$

Corollary 7.13 *Let g_n be the total number of rows in all truth tables for bracketed implications with n variables, then*

$$\lim_{n\to\infty} \frac{k_n^{\#2}}{g_n} = \lim_{n\to\infty} \frac{k_n^{\#3}}{g_n} = \frac{2-\sqrt{2}}{4}, \quad \lim_{n\to\infty} \frac{k_n^{\#1}}{g_n} = \frac{\sqrt{2}}{2}, \quad \lim_{n\to\infty} \frac{h_n}{g_n} = 0.$$

Corollary 7.14

$$\lim_{n\to\infty} \frac{k_n^{\#2}}{k_n^{\#1}} = \frac{\sqrt{2}-1}{2}, \quad \lim_{n\to\infty} \frac{k_n^{\#1}}{k_n^{\#2}} = 2\sqrt{2}+2.$$

8 Parity

Recall the recurrence relation of g_n

$$g_n = \sum_{i=1}^{n-1} g_i g_{n-i}$$

Let a_i and b_i be the corresponding number of false, and truth entries in the considered truth table then:

$$g_n = \sum_{i=1}^{n-1}(a_i+b_i)(a_{n-i}+b_{n-i}) = \sum_{i=1}^{n-1} a_i a_{n-i} + \sum_{i=1}^{n-1} a_i b_{n-i} + \sum_{i=1}^{n-1} b_i a_{n-i} + \sum_{i=1}^{n-1} b_i b_{n-i} \tag{20}$$

Let

$$z_n^{\#1} = \sum_{i=1}^{n-1} a_i a_{n-i}, \quad z_n^{\#2} = \sum_{i=1}^{n-1} a_i b_{n-i}, \quad z_n^{\#3} = \sum_{i=1}^{n-1} b_i a_{n-i}, \quad z_n^{\#4} = \sum_{i=1}^{n-1} b_i b_{n-i}.$$

Theorem 8.1 *For $i = 1,2,3,4$ each sequence $z_n^{\#i}$ preserves the parity of Catalan numbers.*

Proof Since $a_i = (2^i C_i - b_i)$ then

$$z_n^{\#1} = \sum_{i=1}^{n-1}(2^i C_i - b_i)(2^{n-i} C_{n-i} - b_{n-i}), \quad z_n^{\#2} = \sum_{i=1}^{n-1}(2^i C_i - b_i) b_{n-i},$$

$$z_n^{\#3} = \sum_{i=1}^{n-1} b_i (2^{n-i} C_{n-i} - b_{n-i}), \quad z_n^{\#4} = \sum_{i=1}^{n-1} b_i b_{n-i}.$$

If an additive partition of $z_n^{\#1}$, is odd, then it comes as a pair; i.e.

$$(2^iC_i - b_i)(2^{n-i}C_{n-i} - b_{n-i}) \in \mathbb{O} \iff b_i, b_{n-i} \in \mathbb{O}.$$

Hence, $\Big((2^iC_i - b_i)(2^{n-i}C_{n-i} - b_{n-i}) + (2^{n-i}C_{n-i} - b_{n-i})(2^iC_i - b_i)\Big) \in \mathbb{E}$.

Thus, $z_n^{\#1}$ can be expressed as a piecewise function depending on the parity of n:

$$z_n^{\#1} = \begin{cases} 2\sum_{i=1}^{\frac{n-1}{2}}((2^iC_i - b_i)(2^{n-i}C_{n-i} - b_{n-i})) & \text{if } n \in \mathbb{O}, \\ \left(2\sum_{i=1}^{\frac{n-2}{2}}((2^iC_i - b_i)(2^{n-i}C_{n-i} - b_{n-i}))\right) + (2^{\frac{n}{2}}C_{\frac{n}{2}} - b_{\frac{n}{2}})^2 & \text{if } n \in \mathbb{E}. \end{cases}$$

Finally,

$$z_n^{\#1} \in \mathbb{O} \iff (2^{\frac{n}{2}}C_{\frac{n}{2}} - b_{\frac{n}{2}})^2 \in \mathbb{O} \iff b_{\frac{n}{2}} \in \mathbb{O} \iff n = 2^i, \ \ \forall i \in \mathbb{N}.$$

Similarly, if an additive partition of $z_n^{\#2}$, is odd, then it comes as a pair; i.e.

$$(2^iC_i - b_i)b_{n-i} \in \mathbb{O} \iff b_i, b_{n-i} \in \mathbb{O} \iff (2^{n-i}C_{n-i} - b_{n-i})b_i \in \mathbb{O}.$$

Hence, $\Big((2^iC_i - b_i)f_{n-i} + (2^{n-i}C_{n-i} - b_{n-i})b_i\Big) \in \mathbb{E}$.
Thus, $z_n^{\#2}n$ can be expressed as a piecewise function depending on the parity of n:

$$z_n^{\#2} = \begin{cases} \sum_{i=1}^{\frac{n-1}{2}}((2^iC_i - b_i)b_{n-i} + (2^{n-i}C_{n-i} - b_{n-i})b_i) & \text{if } n \in \mathbb{O}, \\ \left(\sum_{i=1}^{\frac{n-2}{2}}((2^iC_i - b_i)b_{n-i} + (2^{n-i}C_{n-i} - b_{n-i})b_i)\right) + (2^{\frac{n}{2}}C_{\frac{n}{2}} - b_{\frac{n}{2}})b_{\frac{n}{2}} & \text{if } n \in \mathbb{E}. \end{cases}$$

Finally,

$$z_n^{\#2} \in \mathbb{O} \iff (2^{\frac{n}{2}}C_{\frac{n}{2}} - b_{\frac{n}{2}})b_{\frac{n}{2}} \in \mathbb{O} \iff b_{\frac{n}{2}} \in \mathbb{O} \iff n = 2^i, \ \ \forall i \in \mathbb{N}.$$

The sequences $z_n^{\#3}$ preserves the parity of Catalan numbers, observe that

$$z_n^{\#3} = \sum_{i=1}^{n-1} b_i(2^{n-1}C_{n-i} - b_{n-i}) = \sum_{i=1}^{n-1}(2^iC_i - b_i)b_{n-i} = z_n^{\#2}.$$

The sequence $z_n^{\#4}$ preserves the parity of the Catalan numbers, since

$$z_n^{\#4} = \begin{cases} 2(b_1b_{n-1} + b_2b_{n-2} + \ldots + b_{\frac{n-1}{2}}b_{\frac{n+1}{2}}) & \text{if } n \in \mathbb{O}, \\ 2(b_1b_{n-1} + b_2b_{n-2} + \ldots + b_{\frac{n-2}{2}}b_{\frac{n+2}{2}}) + b_{\frac{n}{2}}^2 & \text{if } n \in \mathbb{E}. \end{cases}$$

For $n \geq 2$

$$z_n^{\#4} \in \mathbb{O} \iff b_{\frac{n}{2}}^2 \in \mathbb{O} \iff b_{\frac{n}{2}} \in \mathbb{O} \iff n = 2^i \ \ \forall i \in \mathbb{N}.$$

⋆

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Ne yazık, pişmiş ekmek çiğlerin elinde;
Ne yazık, çeşmeler cimrilerin elinde.
O canım güzeller güzeli kömür gözleriyle,
Çakaların, uğruların, eğrilerin elinde.

Ö. Hayyam